

QUANTUM EFFECTS IN A ROTATING SPACETIME

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Abstract

The behavior of a arbitrary coupled quantum scalar field is studied in the background of the Gödel spacetime. Closed forms are derived for the effective action and the vacuum expectation value of quadratic field fluctuations by using ζ -function regularization. Based on these results, we argue that causality violation presented in this spacetime can not be removed by quantum effects.

1 Introduction and motivation

Whether or not the laws of physics permit the existence of closed timelike curves (CTCs) is one the important problems in the research field of modern theoretical physics. A number of familiar spacetimes make it clear that general relativity, as it is normally formulated, does not exclude the violation of causality in large scale, despite its local Lorentzian character.

The Gödel model is not the first but perhaps the best known example of a solution of Einstein's field equations in which causality may be violated and became a paradigm for causality violations in gravitational theory [1].

The most general form for a Gödel-type homogeneous spacetime was found in 1983 by Rebouças and Tiomno [2]. It can be expressed as the direct Riemannian sum $dz^2 + d\sigma^2$ of a flat factor and the $(2+1)$ -dimensional metric

$$d\sigma^2 = dr^2 + \frac{\sinh^2 mr}{m^2} d\varphi^2 - \left(\frac{4\Omega}{m^2} \sinh^2 \left(\frac{mr}{2} \right) d\varphi + dt \right)^2, \quad (1)$$

where Ω and m are constants; m^2 can be continued to negative values, for $m^2 \rightarrow 0$ obtaining Som-Raychaudhuri spacetime [3]. The solution originally proposed by Gödel corresponds to the case $m^2 = 2\Omega^2$

$$ds^2 = dr^2 + \frac{\sinh^2 \sqrt{2}\Omega r}{2\Omega^2} d\varphi^2 + dz^2 - \left(\frac{2}{\Omega} \sinh^2 \left(\frac{\sqrt{2}\Omega r}{2} \right) d\varphi + dt \right)^2, \quad (2)$$

where $\Omega \geq 0$ is a constant parameter related to the vorticity of the matter, and $-\infty < t, z < \infty$, $0 \leq r < \infty$, $0 < \varphi \leq 2\pi$. The source of this geometry is a perfect fluid with constant density ρ and no pressure ($p = 0$). Einstein's equations with a cosmological constant Λ are satisfied if between Ω , Λ and ρ the following relation holds

$$\Omega^2 = -\Lambda = 4\pi\rho. \quad (3)$$

The study of geodesics showed that this spacetime is geodesically complete (and so singularity free) [4, 5]. Furthermore, because the universe is spacetime homogeneous, there are CTCs through every event [6] (hence the causal violation is not localized to some small region).

In the last decade many authors have studied features of quantum field theory on a spacetime background that contains CTCs but most of the papers are dealing with confined causality violating spacetime. In these, CTCs are confined within some regions and there exist at least one region free of them; the regions with CTCs are separated from the well behaved spacetime by Cauchy horizons.

Surprisingly enough, the study of quantum field theory in Gödel spacetime did not receive enough attention in the literature. Previous works include [7], [8], [9]; the difficulties in the standard formulation of quantum field theory in Gödel universe have been pointed out by Leahy in [10] and consist mainly in the absence of a complete Cauchy surface and in the incompleteness of the mode solutions to the field equations (see also [11]).

Being a highly symmetric homogeneous spacetime, Gödel spacetime is an excellent model to investigate questions of principle related to the quantization of fields propagating on curved background, the interaction with a global vorticity and the issues related to the lack of global hyperbolicity. Even if one's primary interest is in quantum field theory in a confined causality violating spacetime, we hope that, by widening the context to Gödel spacetime, one may achieve a deeper appreciation of the theory. In particular, one may hopes to attain more general features of a quantum field propagating in a nonglobally hyperbolic spacetime, whether or not containing a Cauchy horizon. Of particular interest is the question whether causality violation which occurs in Gödel spacetime implies divergence of vacuum polarization fluctuations and may be removed by quantum effects. Thus it seems important to study the vacuum polarization for different physical fields in this background.

A different reason to study quantum field theory (QFT) in this background emerges from the fact the Gödel model is the archetypal cosmology exhibiting the properties associated with the rotation of the universe. Despite the fact that the cosmological rotation is very small by the present observation, it cannot be completely ruled out, at least in the early universe [12] (see also [13] for a up to date discussion and a large set of references).

The definition of a quantum field theory on this manifold requires some cares; in the following we shall restrict ourselves to the Euclidean approach to the quantum field theory, where the ζ function renormalization technique is available. The appropriate Euclidean section of Gödel spacetime is found to be the Euclideanized static mixmaster universe. We present arguments for the existence of a natural periodicity of the euclidean time, *i.e.* an intrinsic temperature for a quantum field propagating in this background.

The line element (1) has also another interesting property. We start by considering the most general Taub-NUT-anti de Sitter line-element (which can be obtained by analytical continuation of the solutions discussed in [14]) written in the form

$$ds^2 = \frac{d\chi^2}{V(\chi)} + (\chi^2 + n^2)(d\theta^2 + f^2(\theta)d\varphi^2) - V(\chi) (dt + 4nf^2(\theta/2)d\varphi)^2, \quad (4)$$

where

$$V(\chi) = k \left(\frac{\chi^2 - n^2}{\chi^2 + n^2} \right) + \frac{-2m\chi + \frac{1}{l^2}(\chi^4 + 6n^2\chi^2 - 3n^4)}{\chi^2 + n^2}. \quad (5)$$

The discrete parameter k takes the values 1, 0 and -1 and implies the form of the function $f(\theta)$

$$f(\theta) = \begin{cases} \sin \theta, & \text{for } k = 1 \\ \theta, & \text{for } k = 0 \\ \sinh \theta, & \text{for } k = -1. \end{cases} \quad (6)$$

Here m is the mass parameter, χ a radial coordinate and n the nut charge. For vanishing nut charge, the solutions correspond to spherically symmetric ($k = 1$) or topological ($k = 0, -1$) vacuum black holes, with a negative cosmological constant $\Lambda = -3/l^2$. A hypersurface of constant large radius χ in the four-dimensional Taub-Nut-AdS spacetime has a metric which is proportional to the three-dimensional line element (1) after the identifications $r = \theta l$; $m = 1/l$; $n = \Omega/m^2$.

The Maldacena conjecture [15] implies that the thermodynamics of a quantum gravity with a negative cosmological constant can be modeled by the large N thermodynamics of quantum field theory. Therefore the interest in field quantization in the background (1).

The paper is organized as follows: some problems implied by the Euclidean approach are discussed in section 2 while in section 3 we compute the local ζ -function. The one loop effective action and the vacuum expectation value of the field fluctuations are computed in section 4. We end drawing some conclusions in section 5.

2 The Euclidean section

Usually there is no well-defined quantum field theory in a spacetime containing CTCs. One way to circumvent some of difficulties of working in spacetime which violate causality, is to use Hawking's Euclidean quantization procedure [16]. This approach can be used if some Lorentzian (-causality violating) space has an appropriate Euclidean analytic continuation. CTCs do not exist in Euclidean space, so one can define a field theory on the Euclidean section, and then analytically continue to obtain the result valid for the acausal spacetime. Note that, for a rotating spacetime, the Wick rotation is more problematic and generally involves the analytic continuation of further parameters than the time coordinate.

The use of this approach for Gödel spacetime has been originally suggested by Leahy [10]. However, since the Gödel spacetime is rotating, the correct Euclidean formulation of a field theory is not immediate and the choice of the Euclidean section on which to work is ambiguous.

In Ref.[9], the following line element has been proposed as describing a rotating gravitational instanton

$$ds^2 = dr^2 + \frac{\sin^2 \sqrt{2}\Omega r}{2\Omega^2} d\varphi^2 + dz^2 + \left(-\frac{2}{\Omega} \sin^2\left(\frac{\sqrt{2}\Omega r}{2}\right) d\varphi + dt\right)^2. \quad (7)$$

The source of this geometry is a perfect fluid with density ρ , no pressure ($p = 0$) and

$$\Omega^2 = \Lambda = 4\pi\rho. \quad (8)$$

One recovers the Lorentzian Gödel solution by analytically continuing $\Omega \rightarrow i\Omega$ and $t \rightarrow -it$. Note the analogy with the case of rotating black hole where a real Euclidean metric is usually obtained by supplementing the analytic continuation $t \rightarrow it$ of the time coordinate t by a further transformation $J \rightarrow iJ$ (where J is the real angular momentum [17]). However, in contrast with asymptotic meaning of rotation and other quantities of a rotating black hole, in this case the rotation has a well defined local character being related to the matter contents of the universe.

Thus we always perform the calculation in the Euclidean geometry; analytically continuing at the end of the calculation will yield the results for the acausal Lorentzian spacetime.

In order to clarify the relation between the Lorentzian and Euclidean sections we use a result of Rooman and Spindel who clarified the problem of the global embedding of the Lorentzian section of the Gödel spacetime [18].

Let $(Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, Z^7, Z^8)$ be global complex coordinates on C^8 and let C^8 be endowed with flat metric

$$ds^2 = (dZ^1)^2 + (dZ^2)^2 + (dZ^3)^2 + (dZ^4)^2 + (dZ^5)^2 - (dZ^6)^2 - (dZ^7)^2 - (dZ^8)^2. \quad (9)$$

We define the complexified Gödel spacetime M^C as the algebraic variety in C^8 determined by the four polynomials

$$\begin{aligned} -(Z^1)^2 - (Z^2)^2 + (Z^5)^2 + (Z^6)^2 &= \frac{4}{\Omega^2}, \\ (Z^3)^2 - (Z^7)^2 - (Z^8)^2 &= \frac{1}{2\Omega^2}, \\ Z^7 &= \frac{\Omega}{2\sqrt{2}}(Z^1 Z^5 - Z^2 Z^6), \\ Z^8 &= \frac{\Omega}{2\sqrt{2}}(Z^2 Z^5 + Z^1 Z^6), \end{aligned} \quad (10)$$

where Ω is an complex parameter determined by the matter content.

The Lorentzian and Euclidean sections of interest, denoted by M^L and M^E are the subsets of M^C stabilized by the respective antiholomorphic involutions

$$J_L : (Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, Z^7, Z^8) \rightarrow (\bar{Z}^1, \bar{Z}^2, \bar{Z}^3, \bar{Z}^4, \bar{Z}^5, \bar{Z}^6, \bar{Z}^7, \bar{Z}^8), \quad (11)$$

supplemented by the condition $\Omega \rightarrow \bar{\Omega}$, and

$$J_E : (Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, Z^7, Z^8) \rightarrow (\bar{Z}^1, \bar{Z}^2, -\bar{Z}^3, \bar{Z}^4, -\bar{Z}^5, -\bar{Z}^6, \bar{Z}^7, \bar{Z}^8), \quad (12)$$

with $\Omega \rightarrow -\bar{\Omega}$, such that J_L leaves $M^L \subset M$ invariant, $J_L(M^L) = M^L$ while J_E leaves $M^E \subset M$ invariant, $J_E(M^E) = M^E$. Clearly M^L and M^E are real algebraic varieties; on M^L , Z^i are real for all i ; on M^E , Z^i are real for $i = 1, 2, 4, 7, 8$ while Z^3, Z^5 and Z^6 are purely imaginary. The induced metric has a number of different representation depending on the choice of coordinates. An explicit embedding of M^L in terms of the usual Gödel coordinates (which cover the entire variety) reads

$$\begin{aligned} Z^1 &= \frac{2}{\Omega} \sinh \frac{\sqrt{2}\Omega r}{2} \cos(\varphi - \frac{\Omega t}{2}), \\ Z^2 &= \frac{2}{\Omega} \sinh \frac{\sqrt{2}\Omega r}{2} \sin(\varphi - \frac{\Omega t}{2}), \end{aligned}$$

$$\begin{aligned}
Z^3 &= \frac{1}{\sqrt{2}\Omega} \cosh \sqrt{2}\Omega r, \\
Z^4 &= z, \\
Z^5 &= \frac{2}{\Omega} \cosh \frac{\sqrt{2}\Omega r}{2} \cos\left(\frac{\Omega t}{2}\right), \\
Z^6 &= \frac{2}{\Omega} \cosh \frac{\sqrt{2}\Omega r}{2} \sin\left(\frac{\Omega t}{2}\right), \\
Z^7 &= \frac{2}{\sqrt{2}\Omega} \sinh \frac{\sqrt{2}\Omega r}{2} \cos \varphi, \\
Z^8 &= \frac{2}{\sqrt{2}\Omega} \sinh \frac{\sqrt{2}\Omega r}{2} \sin \varphi,
\end{aligned} \tag{13}$$

while the embedding of M^E is

$$\begin{aligned}
Z^1 &= \frac{2}{\Omega} \sin \frac{\sqrt{2}\Omega r}{2} \cos\left(\varphi - \frac{\Omega t}{2}\right), \\
Z^2 &= \frac{2}{\Omega} \sin \frac{\sqrt{2}\Omega r}{2} \sin\left(\varphi - \frac{\Omega t}{2}\right), \\
Z^3 &= \frac{1}{\sqrt{2}\Omega} \cos \sqrt{2}\Omega r, \\
Z^4 &= z, \\
Z^5 &= \frac{2}{\Omega} \cos \frac{\sqrt{2}\Omega r}{2} \cos\left(\frac{\Omega t}{2}\right), \\
Z^6 &= \frac{2}{\Omega} \cos \frac{\sqrt{2}\Omega r}{2} \sin\left(\frac{\Omega t}{2}\right), \\
Z^7 &= \frac{2}{\sqrt{2}\Omega} \sin \frac{\sqrt{2}\Omega r}{2} \cos \varphi, \\
Z^8 &= \frac{2}{\sqrt{2}\Omega} \sin \frac{\sqrt{2}\Omega r}{2} \sin \varphi,
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
(Z^1)^2 + (Z^2)^2 + (Z^5)^2 + (Z^6)^2 &= \frac{4}{\Omega^2}, \\
(Z^3)^2 + (Z^7)^2 + (Z^8)^2 &= \frac{1}{2\Omega^2}, \\
Z^7 &= \frac{\Omega}{2\sqrt{2}}(Z^1 Z^5 - Z^2 Z^6), \\
Z^8 &= \frac{\Omega}{2\sqrt{2}}(Z^2 Z^5 + Z^1 Z^6).
\end{aligned} \tag{15}$$

It follows that M^L does not admit a global foliation with spacelike hypersurfaces [18]; the isometries of M^C are clear from the above construction. On M^E the globally-defined Killing vector ∂_t generates a $U(1)$ isometry group of rotations; thus the Euclidean Gödel time t is periodic with a period $4\pi/\Omega$. A periodicity $\beta \neq \frac{4\pi}{\Omega}$ of the Euclidean coordinate t implies a Dirac's delta singularity in the curvature of the manifold at $r = 0$.

In [9] the line element (2.1) was obtained considering it as the direct sum of the metric g_1 on a 3-dimensional space S^3 of constant curvature, $dl^2 = \gamma_{ab}\sigma^a\sigma^b$ (where the σ^a 's are the basis one-forms on the three sphere satisfying the structure relations $d\sigma^a = \frac{1}{2}\epsilon_{bc}^a\sigma^b\sigma^c$) and the metric g_2 defined by $ds_2^2 = dz^2$ on the 1-dimensional manifold R defined by the coordinate z . Here ϵ_{bc}^a , components of the totally antisymmetric tensor, are the structure constants for the rotation group $SO(3)$ and the $\gamma_{ab} = l_a^2\delta_{ab}$ are constant of the space. Using the Euler angle-parametrization, the basis forms are

$$\begin{aligned}\sigma^1 &= -\sin\Psi d\theta + \cos\Psi \sin\theta d\varphi, \\ \sigma^2 &= -\cos\Psi d\theta + \sin\Psi \sin\theta d\varphi, \\ \sigma^3 &= d\Psi + \cos\theta d\varphi,\end{aligned}\tag{16}$$

with $0 < \theta \leq \pi$, $0 < \varphi \leq 2\pi$, $0 < \Psi \leq 4\pi$. Taking $l_1^2 = l_2^2 = l_3^2/2 = 1/2\Omega^2$, and considering the coordinates transformations $\theta = \sqrt{2}\Omega r$, $t = -\frac{1}{\Omega}(\varphi + \Psi)$ we obtain the line element (3), the coordinate range in order to avoid a Dirac's delta singularity being $0 < \sqrt{2}\Omega r \leq \pi$, $0 < t \leq 4\pi/\Omega$, $0 < \varphi \leq 2\pi$, $-\infty < z < \infty$.

Given the thermodynamical principle that the temperature T is inversely related to the period β , $T = \beta^{-1}$, we might therefore expect that the temperature of a quantum field propagating in Gödel spacetime would be $\Omega/4\pi$ refereed to the Killing vector generated by the Lorentzian time t . Thus it corresponds to the heat bath and will be in a mixed quantum state. A free field would propagate straight through the heat bath and not notice its existence but an interacting field will be affected and will lose quantum coherence to the heat bath.

For both Hawking and Unruh effects, temperature emerges from information loss associated with real and accelerated-observer horizons, respectively. The case of the Gödel spacetime is rather different, given the different global structure. This spacetime does not present a causal horizon that hides information; however there is an information loss associated with CTCs. According to a more general argument presented in [16], one might expect loss of quantum coherence whenever one has CTCs, because there will be a part of the quantum state that one doesn't measure initially or finally.

Also, it is worth remarking that the line element (2.1) corresponds to the Euclidean section of a static Taub universe [19] written in a slightly modified form and some finite temperature expressions in the Gödel case can be in principle read off from the zero temperature static Taub results (a similar relation exists *e.g.* between the QFT in cosmic string spacetime and Rindler spacetime [20]). QFT in Taub spacetime has been discussed in the past by many authors ([21],[22], [23]).

Further interest in this line-element emerges recently from the AdS/CFT correspondence, since the

squashed three sphere is the boundary of a euclidean four dimensional Taub-Nut-AdS spacetime. Recent papers on field quantization on this geometry are [24, 25].

3 Local zeta function

A very convenient method to compute the Euclidean Green function, the vacuum fluctuation and the one-loop renormalized stress tensor for a quantum scalar field propagating in the background (1) is to use the formalism of “the direct local ζ -function approach” [26]. The local ζ -function related to the elliptic operator A positive definite on the Euclidean manifold M^E , can be defined as the analytical continuation of the series

$$\zeta(s, x|A) = \sum'_N \lambda_N^{-s} \Phi_N^*(x) \Phi_N(x), \quad (17)$$

(where ' indicates that possible null eigenvalues are omitted). For a scalar field, the global ζ -function can be obtained by integrating the local ζ -function $\zeta(s, x|A)$

$$\zeta(s|A) = \int_{M^E} d^4x \sqrt{g} \zeta(s, x|A), \quad (18)$$

where, through the spectral representation, $\Phi_N(x)$ is a complete series of normalized eigenvectors of the elliptic differential second-order selfadjoint operator $A = -\nabla^\mu \nabla_\mu + M^2 + \xi R$ with eigenvalues λ_N . Here ξ is a parameter which fixes the coupling of the field to the gravity by means of the scalar curvature R . Actually, when the manifold is non-compact, only the local zeta function has a precise mathematical meaning, since the integration requires the introduction of cutoffs or smearing functions to avoid divergences [26].

The eigenfunctions of the operator A in the Euclideanized Gödel spacetime are just the normalized rotation matrices [9]

$$\Phi_N = \sqrt{\frac{\Omega^3(2J+1)}{8\pi^3}} D_{mm'}^J(\sqrt{2}\Omega r) e^{i((m-m')\varphi + k_z z - m\Omega t)}, \quad (19)$$

$D_{mm'}^J(\sqrt{2}\Omega r)$ being the Wigner functions. The corresponding eigenvalues are

$$\lambda_N = 2\Omega^2 \left(J(J+1) + \frac{k_z^2 + M^2 + \xi R}{2\Omega^2} - \frac{m^2}{2} \right), \quad (20)$$

where $N = J, m, m', k_z$, $-\infty < k_z < \infty$; J takes all values of positive integers and half integers and $m, m' = -J, -J+1, \dots, J-1, J$ (notice λ_N has an m' degeneracy). To compute $\zeta(s, x|A)$ we first use the sum rule

$$\sum_{m''} D_{mm''}^J (D_{m'm''}^J)^* = \delta_{mm'}$$

to obtain

$$\zeta(s, x|A) = \frac{\Omega^4}{(4\pi)^2} \frac{\Gamma(s - \frac{1}{2})}{\sqrt{2\pi}\Gamma(s)} \left(\frac{2}{\Omega^2}\right)^s \sum_{J=0}^{\infty} \sum_{m=-J}^J \frac{(2J+1)}{\left(J(J+1) + \frac{m^2 + \xi R}{2\Omega^2} - \frac{m^2}{2}\right)^{s - \frac{1}{2}}}. \quad (21)$$

Therefore, due to the symmetry of spacetime, the local zeta function do not depend on x and so the corresponding fluctuations of the field. Somewhat similar series occur when discussing quantum effects in Taub spacetime; however, the approach presented here and the final results in this paper differs from those in existing literature on the same subject [21]-[24].

The summation over m can be performed by the expansion

$$\zeta(s, x|A) = \frac{\Omega^4}{(4\pi)^2} \frac{1}{\sqrt{2\pi}\Gamma(s)} \left(\frac{2}{\Omega^2}\right)^s \sum_{k=0}^{\infty} \sum_{J=0}^{\infty} \sum_{m=-J}^J \frac{(-1)^k (2J+1) m^{2k}}{2^k k! \left(J(J+1) + \frac{m^2 + \xi R}{2\Omega^2} - \frac{m^2}{2}\right)^{s+k-\frac{1}{2}}} \frac{\Gamma(-s + \frac{3}{2})}{\Gamma(-s - k + \frac{3}{2})}, \quad (22)$$

and making use of the Plana sum formula which says that, for a function $f(m)$

$$\sum_{m=j}^{\infty} f(m) = \frac{f(j)}{2} + \int_j^{\infty} f(x) dx + i \int_0^{\infty} \frac{dt}{e^{2\pi t} - 1} [f(j + it) - f(j - it)]. \quad (23)$$

We find

$$\begin{aligned} \zeta(s, x|A) = & \frac{\Omega^4}{(4\pi)^2} \frac{\Gamma(s - 1/2)}{\sqrt{2\pi}\Gamma(s)} \left(\frac{2}{\Omega^2}\right)^s \left(\sum_{n=1}^{\infty} \frac{n^2}{(n^2 + \sigma)^{s-1/2}} + \right. \\ & + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k k!} \frac{\Gamma(-s + \frac{3}{2})}{\Gamma(-s - k + \frac{3}{2})} \left[\frac{1}{2k+1} I(s + k - \frac{1}{2}; 2k+1; \sigma) + \right. \\ & \left. \left. + I(s + k - \frac{1}{2}; 2k; \sigma) + \sum_{p=0}^{k-1} \binom{2k}{2p+1} 2^{2k-2p-1} \frac{B_{2k-2p}}{k-p} I(s + k - \frac{1}{2}; 2p+1; \sigma) \right] \right), \end{aligned} \quad (24)$$

where B_k are Bernoulli numbers and $\sigma = 2\Omega^2(M^2 + \zeta R) - 1$ (for a massless conformally coupled scalar field $\sigma = -\frac{1}{3}$; while for $M = 0, \xi = 0, \sigma = -1$; in this case the value $n = 1$ corresponds to a null eigenvalues and is omitted). In (24) we have defined

$$I(q, r, \sigma) = \sum_{n=1}^{\infty} \frac{n(n-1)^r}{(n^2 + \sigma)^q}. \quad (25)$$

Some properties of the functions $I(q, r, \sigma)$ are discussed in the Appendix; they possess poles at $q = m/2 + 1 - n$, where $n = 0, 1, \dots$

The general expression of $I(q, r, \sigma)$ is quite complicated; a more practical approach is to use a low- σ binomial expansion of the kind

$$\sum_{n=1}^{\infty} \frac{n^r}{(n^2 + \sigma)^q} = \sum_{l=0}^{\infty} \frac{\sigma^l}{l!} \frac{\Gamma(-q+1)}{\Gamma(-q-l+1)} \zeta_R(2q + 2l - r), \quad (26)$$

where ζ_R is the usual Riemann zeta-function which can be analytically continued in the whole complex plane except for the only singular point as $s = 1$.

Thus we obtain

$$\begin{aligned}\zeta(s, x|A) = & \frac{\Omega^4}{(4\pi)^2} \frac{1}{\sqrt{2\pi}\Gamma(s)} \left(\frac{2}{\Omega^2} \right)^s \left(\sum_{l=0}^{\infty} \frac{(-1)^l \sigma^l}{l!} \Gamma(s+l-\frac{1}{2}) \zeta_R(2s+2l-3) + \right. \\ & + \sum_{k=1}^{\infty} \frac{1}{2^k k!} \left[\frac{1}{2k+1} F(s+k-\frac{1}{2}; 2k+1; \sigma) + F(s+k-\frac{1}{2}; 2k; \sigma) + \right. \\ & \left. \left. + \sum_{p=0}^{k-1} \binom{2k}{2p+1} 2^{2k-2p-1} \frac{B_{2k-2p}}{k-p} F(s+k-\frac{1}{2}; 2p+1; \sigma) \right] \right),\end{aligned}\quad (27)$$

where we have defined

$$F(s+k-\frac{1}{2}; r; \sigma) = \sum_{l=0}^{\infty} \sum_{m=0}^r \binom{r}{m} \frac{(-1)^{l+m+r} \sigma^l}{l!} \Gamma(s+k+l-\frac{1}{2}) \zeta_R(2s+2k+2l-m-2). \quad (28)$$

The expression (27) involves the divergent terms coming from the poles of $\Gamma(z)$ at $z = -k$ ($k = 0, 1, 2, \dots$)

$$\Gamma(-k+\epsilon) = \frac{(-1)^k}{k!} \left(\frac{1}{\epsilon} - \Psi(k+1) + O(\epsilon) \right), \quad (29)$$

and the pole of $\zeta_R(z)$ at $z = 1$

$$\zeta_R(1+2\epsilon) = \frac{1}{2\epsilon} + \gamma, \quad (30)$$

where γ is Euler-Mascheroni constant, $\gamma = 0.5772157\dots$

The summation above converge whenever $\text{Re } s > 2$ defining an analytical function which can be extended into a meromorphic function defined on the complex s -plane except for two simple poles on the real axis at $s = 1$ and $s = 2$

$$\zeta(s, x|A) = \zeta_{\text{regular}}(s, x|A) + \frac{1}{(4\pi)^2} \left(\frac{1}{s-2} - \frac{1}{2} \left(\sigma + \frac{1}{3} \right) \frac{\Omega^2}{s-1} \right), \quad (31)$$

in concordance with the general theory, since the local Seeley-DeWitt coefficients a_i for Gödel geometry are given by $a_0 = 1$ and $a_1 = -\frac{\Omega^2}{2}(\sigma + \frac{1}{3})$.

Also it can be shown that the local zeta-function satisfies the relation

$$\zeta(0, x|A) = \frac{a_2}{4\pi^2} = \frac{\Omega^2}{(4\pi)^2} \left(\frac{4}{45} + \frac{1}{4} \left(\sigma + \frac{1}{3} \right)^2 \right). \quad (32)$$

In deducing the above expressions we have used the relations

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{2^k k!} &= (\sqrt{2}-1)\sqrt{\pi}, \\ \sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{2^k k!} k &= \frac{\sqrt{2\pi}}{2}, \\ \sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{2^k k!} k^2 &= \frac{5\sqrt{2\pi}}{4}.\end{aligned}\quad (33)$$

The global zeta-function, defined as

$$\zeta(s|A) = \sum_N \lambda_N^{-s} \quad (34)$$

satisfies the relation

$$\zeta(s|A) = \frac{16\pi^3}{\Omega^3} \zeta(s, x|A). \quad (35)$$

4 Some physical applications

Two very useful quantities in the study of quantum effects in curved spacetimes are $\langle \Phi^2 \rangle$ and S_{eff} , where S_{eff} is one loop effective action of the scalar field Φ . $\langle \Phi^2 \rangle$ is a useful quantum quantity because it gives one qualitative information about the renormalized stress tensor $\langle T_k^i \rangle$ and can often be computed with much less effort. $\langle \Phi^2 \rangle$ also provides information about spontaneous symmetry breaking in a given background.

In the path integral approach, the effective action for a scalar field can be formally expressed as the functional determinant of the operator A as

$$S_{eff}[\Phi, g] = -\frac{1}{2} \ln \det\left(\frac{A}{\mu^2}\right), \quad (36)$$

where μ is a scale parameter necessary from dimensional considerations (this parameter may remain in the final results and thus can be reabsorbed into the renormalized gravitational constant as well as other physically measurable parameters).

This determinant is however a formally divergent quantity and needs to be regularized. In the framework of the ζ -function renormalization, the regularized determinant reads [26]

$$S_{eff}[\Phi, g] = -\frac{1}{2} \frac{d}{ds} \Big|_{s=0} \zeta(s| \frac{A}{\mu^2}) = -\frac{1}{2} \frac{d}{ds} \Big|_{s=0} \zeta(s|A) - \frac{1}{2} \frac{d}{ds} \Big|_{s=0} \zeta(0|A) \ln \mu^2. \quad (37)$$

For our application, we are interested in $\zeta'(0|A)$. To compute the derivative of the global zeta-function in $s = 0$, we note that if $f(s)$ is a smooth function of s , then

$$\frac{d}{ds} \left(\frac{f(s)}{\Gamma(s)} \right) \Big|_{s=0} = f(0). \quad (38)$$

By using the relations (27),(35) and the expansion

$$\frac{1}{\Gamma(s)} \sim s + \gamma s^2 + O(s^3)$$

as $s \rightarrow 0$, with a bit of algebra, one obtains

$$S_{eff} = -\frac{1}{2} \left(T + \frac{3\gamma}{2} \frac{a_2}{(4\pi)^2} + \frac{a_2}{(4\pi)^2} \ln \frac{2}{\mu^2 \Omega^2} + \frac{1}{2} \frac{\Omega^4}{(4\pi)^2} \frac{1}{\sqrt{2\pi}} (c_1 + c_2 + c_3 \sigma^2) \right), \quad (39)$$

where

$$\begin{aligned}
T = & \frac{\Omega^4}{(4\pi)^2} \frac{1}{\sqrt{2\pi}} \left(-\frac{\sqrt{\pi}}{240} + \frac{\sqrt{\pi}}{12} \sigma + \sum_{l=3}^{\infty} \frac{(-1)^l \sigma^l}{l!} \Gamma(l - \frac{1}{2}) \zeta_R(2l - 3) + \right. \\
& + \sum_{k=1}^{\infty} \frac{1}{2^k k!} \left[\frac{1}{2k+1} \sum_{l=0}^{\infty} \sum_{\substack{m=0 \\ (l,m) \neq \{M_1\}}}^{2k+1} \binom{2k+1}{m} \frac{(-1)^{l+m+1} \sigma^l}{l!} \Gamma(k+l - \frac{1}{2}) \zeta_R(2k+2l-m-2) \right. \\
& + \sum_{l=0}^{\infty} \sum_{\substack{m=0 \\ (l,m) \neq \{M_2\}}}^{2k} \binom{2k}{m} \frac{(-1)^{l+m} \sigma^l}{l!} \Gamma(k+l - \frac{1}{2}) \zeta_R(2k+2l-m-2) \\
& \left. + \sum_{l=0}^{\infty} \sum_{\substack{p=0 \\ (l,m,p) \neq \{M_3\}}}^{k-1} \sum_{m=0}^{2p+1} \binom{2k}{2p+1} \frac{(-1)^{l+m+1} \sigma^l}{l!} \frac{B_{2k-2p}}{k-p} \Gamma(k+l - \frac{1}{2}) \zeta_R(2k+2l-m-2) \right], \quad (40)
\end{aligned}$$

and $c_1 = 0.310856$; $c_2 = 1.17983$; $c_3 = 0.1721$. Here the indices corresponding to divergent terms are omitted and therefore $M_1 = \{(0, 2k-3), (1, 2k-1), (2, 2k+1)\}$, $M_2 = \{(0, 2k-3), (1, 2k-1)\}$, $M_3 = \{(0, k-2, 2p+1), (0, k-1, 2p-1), (1, k-1, 2p+1)\}$.

The scale μ represents the usual ambiguity due to a remaining finite renormalization and it remains into the final result whenever another fixed scale is already present in the theory. The Newton constant G and $1/\Omega$ determine two different scales in the theory. The constant $1/\Omega$ (related to the scalar curvature via $R = 2\Omega^2$) is a large distance (cosmological) scale while the distance $l_{Pl} \sim G^{1/2}$ is the Planck scale of the theory. As expected, the parameter Ω combines with the renormalization scale μ to leave a dimensionless argument for the logarithm.

The vacuum expectation value of the field fluctuations can be computed within the ζ -function regularization scheme by means of the formula

$$\langle \Phi^2(x) \rangle = \left. \frac{d}{ds} \right|_{s=0} s \zeta(s+1, x|A) + s \zeta(s+1, x|A) |_{s=0} \ln \mu^2, \quad (41)$$

where $\zeta(s, x|a)$ is the local zeta-function and μ is again the renormalization mass scale. Recently it has been shown that this procedure leads to the same results as the point-splitting technique [27].

From the expression (24), (31) the behavior of the local zeta-function and its derivative near $s = 1$ is a simple matter of algebra, and we can easily obtain the expectation value of a field fluctuations

$$\begin{aligned}
\langle \Phi^2 \rangle = & \frac{\Omega^2}{(4\pi)^2} \frac{1}{\sqrt{2\pi}} \left(-\frac{\sqrt{\pi}}{12} + \sum_{l=2}^{\infty} \frac{(-1)^l \sigma^l}{l!} \Gamma(l + \frac{1}{2}) \zeta_R(2l - 1) + \right. \\
& + \sum_{k=1}^{\infty} \frac{1}{2^k k!} \left[\frac{1}{2k+1} \sum_{l=0}^{\infty} \sum_{\substack{m=0 \\ (l,m) \neq \{N_1\}}}^{2k+1} \binom{2k+1}{m} \frac{(-1)^{l+m+1} \sigma^l}{l!} \Gamma(k+l + \frac{1}{2}) \zeta_R(2k+2l-m) \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{l=0}^{\infty} \sum_{\substack{m=0 \\ (l,m) \neq (0,2k-1)}}^{2k} \binom{2k}{m} \frac{(-1)^{l+m} \sigma^l}{l!} \Gamma(k+l+\frac{1}{2}) \zeta_R(2k+2l-m) \\
& + \sum_{l=0}^{\infty} \sum_{\substack{p=0 \\ (l,p,m) \neq (0,k-1,2p+1)}}^{k-1} \sum_{m=0}^{2p+1} \binom{2k}{2p+1} 2^{2k-2p-1} \frac{B_{2k-2p}(-1)^{l+m+1} \sigma^l}{(k-p)l!} \Gamma(k+l+\frac{1}{2}) \zeta_R(2k+2l-m) \Big] \\
& + \frac{3a_1\gamma}{(4\pi)^2} + \frac{a_1}{(4\pi)^2} \log \frac{2\mu^2}{\Omega^2} + \frac{1}{(4\pi)^2} (\alpha_0 + \alpha_1 \sigma), \tag{42}
\end{aligned}$$

where $\alpha_0 = -0.188269$, $\alpha_1 = -0.152411$ and $N_1 = \{(0, 2k-1), (1, 2k+1)\}$. Again, the parameter Ω and the renormalization scale μ combine to leave a dimensionless argument for the logarithm.

In a “super- ζ -regular theory” ($\sigma = -1/3$) the local zeta-functional and its derivative are finite in $s = 1$; the field fluctuations are simply given by the value in $s = 1$ of the local zeta-function and the above formula reduces to the more usual expression also expected from Green function analysis

$$\langle \Phi^2 \rangle = \zeta(1, x|A). \tag{43}$$

Both $\langle \Phi^2 \rangle$ and S_{eff} are well behaved quantities and we can find their values for a given σ by performing the above summation. Although impressive in appearance, these series are quickly convergent in practice.

Now it is quite straightforward to obtain the partition function and then all other thermodynamically quantities for a scalar field. The free energy is related to the canonical partition function by means of equation $F = -\frac{1}{\beta} \ln Z$. The entropy and the internal energy of the system are given by the usual thermodynamical formula $U = \langle E \rangle = \frac{\partial}{\partial \beta} (\beta F)$ and $S = \beta^2 \frac{\partial}{\partial \beta} F$. In this way it is possible to obtain exact results beyond the WKB approximation discussed in [28]. After all calculation with the Euclideanized Gödel universe have been completed, we analytically continue the results back to real values of Ω , obtaining results valid on the Lorentzian section.

5 Final remarks. Conclusions

In this paper we have obtained exact expressions at one loop for the effective action and the vacuum expectation values for a scalar field propagating in Gödel spacetime. Our expression hold for massless as well as massive fields, with an arbitrary coupling with the scalar curvature. Similar analysis can be carried out for higher spin fields.

The computation presented here is only the first step to the study of physically more interesting effects. Once $\langle \Phi^2 \rangle$ is known it is easy to compute $\langle T \rangle$ (the expectation value of the trace of the stress-energy

tensor) for a conformally coupled massive scalar field. Since $T = -M^2 < \Phi^2 >$ for this field and $< \Phi^2 >$ is a constant, the renormalized value of $< T >$ is just the trace anomaly plus $< \Phi^2 >$, *i.e.*

$$< T > = \frac{a_2}{(4\pi)^2} - M^2 < \Phi^2 >. \quad (44)$$

Given the high symmetry of the manifold, it is very convenient to compute the renormalized stress energy tensor by using the same formalism of “the direct local zeta-function approach” within the one loop renormalized stress-energy tensor of a scalar field is [26]

$$\begin{aligned} < T_{ab}(x) > = \{ \zeta_{ab}(s+1, x|A) + \frac{1}{2} g_{ab} \zeta(s, x|A) + \\ + s [\zeta'_{ab}(s+1, x|A + \ln(\mu^2)) \zeta_{ab}(s+1, x|A)] \} |_{s=0}. \end{aligned} \quad (45)$$

The tensorial zeta-function $\zeta(s, x|A)_{ab}$ is obtained through the s -analytic continuation in the whole complex plane of the series

$$\zeta_{ab}(s, x|A) = \bar{\zeta}_{ab}(s, x|A) + \xi R_{ab}(x) \zeta(s, x|A) - \frac{1}{2} \zeta(s-1, x|A) \quad (46)$$

with

$$\bar{\zeta}_{ab}(s, x|A) = \sum_N \lambda_N^{-s} \nabla_a \Phi_N^* \nabla_b \Phi_N(x), \quad (47)$$

where the homogeneity of the background was taken into account. By using the properties of Wigner functions, it is possible to find the renormalized $\zeta(s, x|A)_{ab}$. Since the calculations are very difficult, the explicit computation of $< T_k^i >$ will be considered elsewhere. The zz component of the renormalized stress tensor is the only one that can be easily computed by taking into account the relation

$$\bar{\zeta}_{zz}(s, x|A) = \frac{\Omega^2}{2(s-1)} \zeta(s-1, x|A). \quad (48)$$

Hence

$$< T_{zz}(x) > = \frac{1}{2} \left(\frac{d}{ds} \Big|_{s=0} \zeta(s, x|A) + \zeta(0, x|A) \ln(\mu^2) \right). \quad (49)$$

However, few remarks can be given even at this stage. In general, it seems that divergences in the energy momentum-tensor occur when one has closed or self-interacting null geodesics [29]. However, in Gödel universe there is no Cauchy horizon containing closed null geodesics. Also from the analysis of free motion in Gödel spacetime, we know that the geodesic motion do not follow a CTC [4, 5].

We find that both the vacuum fluctuations of the field and the renormalized stress energy-tensor $< T_k^i >$ are well behaved quantities in every spacetime point. Thus CTCs do not mean that the energy-momentum tensor must diverge and we cannot hope the causality violation which occurs in Gödel spacetime might be

removed by quantum effects. A complete answer is not possible until the investigation of the problem of back reaction on the metric.

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6 Appendix. Definition and evaluation of the $I(q, p, \sigma)$ function

In this section we shall study the function $I(q, r, \sigma)$

$$I(q, r, \sigma) = \sum_{n=1}^{\infty} \frac{n(n-1)^r}{(n^2 + \sigma)^q}. \quad (50)$$

This series is a generalization of the one examined by Shen et al. [22]. Following the same procedure, we find that $I(q, r, \sigma)$ have poles at $r = \frac{m}{2} - n + 1$. Employing the Plana sum formula $I(q, r, \sigma)$ can be written as

$$I(q, r, \sigma) = \sum_{m=0}^r \binom{r}{m} (-1)^{r+m} \int_0^{\infty} dx \frac{x^{m+1}}{(x^2 + \sigma)^q} - \int_0^1 dx \frac{x(x-1)^r}{(x^2 + \sigma)^q} + \quad (51)$$

$$+ i^{r+1} \int_0^{\infty} dt \frac{t^r}{e^{2\pi t} - 1} \left(\frac{1+it}{((1+it)^2 + \sigma)^q} + \frac{(-1)^{r+1}(1-it)}{((1+it)^2 + \sigma)^q} \right).$$

The integral

$$\int_0^{\infty} dx \frac{x^{m+1}}{(x^2 + \sigma)^q} = \frac{\sigma^{\frac{m}{2}+1-q}}{2} \frac{\Gamma(\frac{m}{2}+1)\Gamma(q-\frac{m}{2}-1)}{\Gamma(q)} \quad (52)$$

vanishes at negative integer q , but diverges at $q = \frac{m}{2} + 1 - n$, where $n = 0, 1, 2, \dots$

This is the only divergent part on the right hand side of eq. (51). One can expand the singular term in a series about the pole of the Gamma function at $q = \frac{m}{2} + 1 - n$

$$\int_0^{\infty} dx \frac{x^{m+1}}{(x^2 + \sigma)^q} = \frac{\sigma^n}{2} \frac{\Gamma(\frac{m}{2}+1)}{\Gamma(\frac{m}{2}+1-n)} \left[\frac{(-1)^n}{n!} \frac{1}{\nu} + \Psi(n+1) + \frac{1}{2}\nu \left(\frac{\pi^2}{3} + \Psi^2(n+1) - \Psi'(n+1) \right) \right] \quad (53)$$

where $\Psi(x) = \frac{d}{dx} \ln \Gamma(x)$. We denote the singular part of the function $I(q, r, \sigma)$ by I_{-1} and the regular part by I_0 . Therefore

$$I_{-1}(n, r, \sigma) = \sum_{m=0}^r \binom{r}{m} (-1)^{r+m} \frac{\sigma^n}{2} \frac{\Gamma(\frac{m}{2}+1)}{\Gamma(\frac{m}{2}+1-n)} \frac{(-1)^n}{n!},$$

$$\begin{aligned}
I_0(n, r, \sigma) &= \sum_{m=0}^r \left[\binom{r}{m} (-1)^{r+m} \frac{\sigma^n}{2} \frac{\Gamma(\frac{m}{2} + 1)}{\Gamma(\frac{m}{2} + 1 - n)} \Psi(n+1) - \int_0^1 dx \frac{x(x-1)^r}{(x^2 + \sigma)^{\frac{m}{2} + 1 - n}} \right. \\
&\quad \left. + i^{r+1} \int_0^\infty dt \frac{t^r}{e^{2\pi t} - 1} \left(\frac{1+it}{((1+it)^2 + \sigma)^{\frac{m}{2} + 1 - n}} + \frac{(-1)^{r+1}(1-it)}{((1-it)^2 + \sigma)^{\frac{m}{2} + 1 - n}} \right) \right]. \quad (54)
\end{aligned}$$

An analytic result for $I_0(n, r, \sigma)$, $I_{-1}(n, r, \sigma)$ is not accessible; however we may seek the asymptotic expansions for particular ranges of σ . Note that for $\sigma = 1$, the $n = 1$ term is excluded from the above summation.

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